

Toward classification of rational vertex operator algebras with central charges less than 1

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Abstract

The rational and C_2 -cofinite simple vertex operator algebras whose effective central charges and the central charges c are equal and less than 1 are classified. Such a vertex operator algebra is zero if $\tilde{c} < 0$ and \mathbb{C} if $\tilde{c} = 0$. If $\tilde{c} > 0$, it is an extension of discrete Virasoro vertex operator algebra $L(c_{p,q}, 0)$ by its irreducible modules. It is also proved that for any rational and C_2 -cofinite simple vertex operator algebra with $c = \tilde{c}$, the vertex operator subalgebra generated by the Virasoro vector is simple.

1 Introduction

One of the most important problems in the theory of vertex operator algebra is to classify the rational vertex operator algebras. It is not realistic to achieve this goal at this stage due to the limited knowledge of the structure theory and representation theory. If a vertex operator algebra is rational then the central charge c and effective central charge \tilde{c} are rational (cf. [AM], [DLM2]). While the central charge can be negative, the effective central charge is always nonnegative [DM2]. In this paper we classify the rational vertex operator algebras with $c = \tilde{c} < 1$ although we cannot write down the results explicitly.

It is well known that one can construct vertex operator algebras associated to highest weight modules for the Virasoro algebra [FZ]. In particular, each irreducible highest weight module $L(c, 0)$ for any complex number c is a simple vertex operator algebra. Moreover, $L(c, 0)$ is rational if and only if $c = c_{p,q} = 1 - 6(p-q)^2/pq$ for coprime positive integers p, q with $1 < p, q$ [W]. Furthermore, $L(c, 0)$ is unitary if and only if $c = c_{p,p+1}$ for all $p > 1$ or $c \geq 1$ (see [FQS] and [GKO]). Our classification result says that any simple, rational and C_2 -cofinite vertex operator algebra with $c = \tilde{c} < 1$ is an extension of the Virasoro vertex operator algebra $L(c_{p,q}, 0)$ for some p, q . That is, such vertex operator algebra is a finite direct sum of irreducible $L(c_{p,q}, 0)$ -modules.

The main ideal is to use the modular invariance of the q -characters of the modules (see [Z] and [DLM2]) to control the growth of the graded dimensions of the vertex operator algebra. The same idea has been used to classify the holomorphic vertex operator algebras with small central charges [DM1], to prove the nonnegative property of the effective central charges [DM2], and to obtain some uniqueness result on the moonshine vertex operator

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algebra V^\natural [DGL]. The modular invariance property of the q -characters of the modules is also the reason why we use the effective central charges instead of central charges (see Lemma 2.1 below). We should point out that we do not assume that the vertex operator algebra is a unitary representation for the Virasoro algebra and that $\sum_i |\chi_i(q)|^2$ is a modular function over the full modular group where $\chi_i(q)$ are the q -characters of the irreducible modules of the vertex operator algebra (see Section 2).

As a corollary of the main result we prove that for any simple, rational and C_2 -cofinite vertex operator algebra with $c = \tilde{c}$, the vertex operator subalgebra generated by the Virasoro vector is simple. That is, the vertex operator subalgebra is an irreducible highest weight module for the Virasoro algebra.

It is worthy to mention that we do not have a explicit list of such vertex operator algebras. An eventual classification requires to construct all extensions of $L(c_{p,q}, 0)$ for all p, q . In the case that $c = c_{p,p+1}$, the extensions of $L(c_{p,p+1}, 0)$ have been classified in the theory of conformal nets (an analytical approach to conformal field theory) [KL] (also see [X]). Although it is believed that such classification result is valid in the theory of vertex operator algebra, most of such extensions have not been constructed in the context of vertex operator algebra except for a few examples from the code vertex operator algebras and lattice vertex operator algebras.

2 Rational vertex operator algebras

In this section, we review some basic facts on the q -characters of modules for a rational vertex operator algebra. The main feature of these functions is the modular invariance property [Z], and its connection with the vector-valued modular forms [KM]. This connection is the key for us to estimate the growth of the graded dimensions of the vertex operator algebra and its modules. We will also discuss the effective central charge \tilde{c} .

We assume that vertex operator algebra V is simple and is of CFT type. That is,

$$V = \bigoplus_{n=0}^{\infty} V_n \quad (2.1)$$

moreover V_0 is spanned by the vacuum vector $\mathbf{1}$. Following [DLM1], V is called rational if the admissible module category is semisimple. V is called C_2 -cofinite if $V/C_2(V)$ is finite dimensional [Z] where $C_2(V) = \langle u_{-2}v | u, v \in V \rangle$.

A rational vertex operator algebra V has only finitely many irreducible modules $V = M^1, M^2, \dots, M^r$ up to isomorphism such that

$$M^i = \bigoplus_{n \geq 0} M_{\lambda_i + n}^i$$

where λ_i is a rational number and $M_{\lambda_i}^i \neq 0$ (see [DLM1], [DLM2]). Moreover each homogeneous subspace $M_{\lambda_i + n}^i$ is finite dimensional. Let λ_{min} be the minimum among the λ_i . The effective central charge \tilde{c} which appeared in the physics literature [GN] is defined by $\tilde{c} = c - 24\lambda_{min}$. One of the main results in [DM2] is that \tilde{c} is nonnegative, and $\tilde{c} = 0$ if and only if $V = \mathbb{C}$ is trivial.

For each i we define the q -character of M^i as

$$\chi_i(q) = ch_q M^i = \text{tr}_{M^i} q^{L(0)-c/24} = \sum_{n \geq 0} (\dim M_{n+\lambda_i}^i) q^{n-c/24}.$$

It is proved in [Z] (also see [DLM2]) that if V is rational and C_2 -cofinite then each $\chi_i(q)$ is a holomorphic function on the upper half plane \mathbb{H} where $q = e^{2\pi i\tau}$ and the span of these functions affords a representation of the modular group $SL(2, \mathbb{Z})$. For short we also write $\chi_i(\tau)$ for $\chi_i(q)$. Then there exists a group homomorphism ρ from $SL(2, \mathbb{Z})$ to $GL(r, \mathbb{C})$ such that for any $\gamma \in SL(2, \mathbb{Z})$,

$$\chi_i(\gamma\tau) = \sum_j \gamma_{ij} \chi_j(\tau)$$

where $\rho(\gamma) = (\gamma_{ij})$. This exactly says that $\chi(\tau) = (\chi_1(\tau), \dots, \chi_r(\tau))$ is a (meromorphic) vector-valued modular function [KM].

Recall the Dedekind eta function

$$\eta(\tau) = q^{1/24} \phi(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

and the expansion

$$\frac{1}{\prod_{n \geq 1} (1 - q^n)} = \sum_{n=0}^{\infty} p(n) q^n$$

where $p(n)$ is the usual unrestricted partition function. An asymptotic expression for $p(n)$ is given by

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$

as $n \rightarrow \infty$ (cf. [A]). It is clear that $p(n)$ grows faster than n^α for any fixed real number α .

The $\eta(\tau)$ is a modular form of weight $1/2$. Since $\eta(\tau)^{\tilde{c}} \chi_i(\tau)$ is holomorphic at $\tau = i\infty$,

$$\eta(\tau)^{\tilde{c}} \chi(\tau) = (\eta(\tau)^{\tilde{c}} \chi_1(\tau), \dots, \eta(\tau)^{\tilde{c}} \chi_r(\tau))$$

is a holomorphic vector-valued modular form of weight $\tilde{c}/2$. From [KM] the Fourier coefficients a_n of a component of a holomorphic vector-valued modular form satisfy the growth condition $a_n = O(n^\alpha)$ for a constant α independent of n . As a result, we see that

Lemma 2.1 *The Fourier coefficients of each component of $\eta(\tau)^{\tilde{c}} \chi(\tau)$ satisfy a polynomial growth condition $a_n = O(n^\alpha)$.*

3 Vertex operator algebras with $c < 1$

In this section we will prove the main result. We assume that V is a rational and C_2 -cofinite vertex operator algebra with $c = \tilde{c} < 1$. It is proved in [DM2] that \tilde{c} is always nonnegative. If $c = 0$ then $V = \mathbb{C}$. So we assume that $c > 0$.

We first need information on the vertex operator algebras associated to the highest weight modules for the Virasoro algebra (see [FF], [FZ], [W]). We use the standard basis $\{L_n, C | n \in \mathbb{Z}\}$ for the Virasoro algebra. For any two complex numbers c, h we denote the Verma module with central charge c and highest weight h by $V(c, h)$, as usual. Let $\bar{V}(c, 0)$ be the quotient of $V(c, 0)$ modulo submodule generated by $L_{-1}v$ where v a nonzero highest weight vector of $V(c, 0)$ with highest weight 0. We use $L(c, h)$ to denote the irreducible quotient of $V(c, h)$.

We have already defined the q -character $\text{ch}_q M$ if M is a module for any vertex operator algebra. We now extend this definition to any module for the Virasoro algebra with finite dimensional homogeneous subspaces and central charge c . In general the q -character is just a formal power series in q . Note that $\text{ch}_q \bar{V}(c, 0) = \frac{q^{-c/24}}{\prod_{n>1}(1-q^n)}$ and its coefficients grow faster than n^α for any fixed real number α .

Lemma 3.1 *For any $\mu > 0$ the coefficients of $\frac{1}{\prod_{n>1}(1-q^n)^\mu}$ grow faster than any polynomial n^α .*

Proof: Observe that the coefficients $\binom{-\mu}{i}(-1)^i$ of q^i in the expansion of $(1-q)^{-\mu}$ is always positive for any $\mu > 0$. Assume that the coefficients of

$$\frac{1}{\prod_{n>1}(1-q^n)^\mu} = \sum_{n \geq 0} a_n q^n$$

satisfy the polynomial growth condition. Then the coefficients of $\frac{1}{\prod_{n>1}(1-q^n)^{k\mu}}$ satisfy the polynomial growth condition for any positive integer k . But if k is large enough then $k\mu > 1$ and the coefficients of $\frac{1}{\prod_{n>1}(1-q^n)^{k\mu}}$ grow faster than n^α for any real number α . This is a contradiction. \square

Here are some basic facts about these modules. [FF], [FQS],[GKO], [FZ], [W].

Proposition 3.2 *Let c be a complex number. Then the following hold:*

- (i) $\bar{V}(c, 0)$ is a vertex operator algebra and $L(c, 0)$ is a simple vertex operator algebra.
- (ii) The following are equivalent: (a) $\bar{V}(c, 0) = L(c, 0)$, (b) $c \neq c_{p,q} = 1 - 6(p-q)^2/pq$ for all coprime positive integers p, q with $1 < p < q$, (c) $L(c, 0)$ is not rational. In this case, the q -graded character of $L(c, 0)$ is equal to $\frac{q^{-c/24}}{\prod_{n>1}(1-q^n)}$ and its coefficients grow faster than any polynomial in n .
- (iii) The following are equivalent: (a) $\bar{V}(c, 0) \neq L(c, 0)$, (b) $c = c_{p,q}$ for some p, q , (c) $L(c, 0)$ is rational.

We now back to vertex operator algebra V . Let $U = \langle \omega \rangle$ be the vertex operator subalgebra of V . Then there are two possibilities: either U is isomorphic to $\bar{V}(c, 0)$ or $L(c, 0)$ from the structure theory for these modules [FF].

The following is the key lemma.

Lemma 3.3 Assume that $c < 1$. Then the q character $ch_q U$ of U is different from $\frac{q^{-c/24}}{\prod_{n>1}(1-q^n)}$.

Proof: We prove by contradiction. Suppose that $ch_q U$ is equal to $\frac{q^{-c/24}}{\prod_{n>1}(1-q^n)}$. Then

$$\eta(q)^c ch_q U = \frac{(1-q)^c}{\prod_{n>1}(1-q^n)^{1-c}}.$$

By Lemma 3.1 the coefficients of $\frac{1}{\prod_{n>1}(1-q^n)^{1-c}}$ grow faster than any polynomial in n . Set

$$\eta(q)^c ch_q V = \sum_{n \geq 0} b_n q^n.$$

By Lemma 2.1, the coefficients b_n satisfy polynomial growth condition.

Let $f(q) = (1-q)^{-1} \eta(q)^c ch_q V = \sum_{n \geq 0} c_n q^n$. Then

$$c_n = \sum_{i=0}^n b_i$$

for all $n \geq 0$. Since b_n satisfy polynomial growth condition, there exist positive constants C and α such that $|b_n| \leq Cn^\alpha$ for all n . As a result, $|c_n| \leq C(n+1)n^\alpha \leq 2Cn^{\alpha+1}$ for $n > 0$. That is, the coefficients c_n also satisfy the polynomial growth condition.

Since U is a subspace of V we see that $\dim U_n \leq \dim V_n$ for all n . This implies that $ch_q U \leq ch_q V$ and $\eta(q)^c ch_q U \leq \eta(q)^c ch_q V$ as real numbers for any $q \in (0, 1)$. Thus $\frac{(1-q)^c}{\prod_{n>0}(1-q^n)^{1-c}} \leq f(q)$ for all $q \in (0, 1)$. But this is impossible as the coefficients of $\frac{1}{\prod_{n>1}(1-q^n)^{1-c}}$ grow faster than any polynomial in n . The proof is complete. \square

Corollary 3.4 Let V be a simple, rational and C_2 -cofinite vertex operator algebra with central charge $c = \tilde{c} < 1$. Then the vertex operator algebra U generated by the Virasoro element ω is simple and $c = c_{p,q}$ for some coprime p, q such that $1 < p < q$.

Proof: If $c \neq c_{p,q}$ then $\bar{V}(c, 0)$ is irreducible module for the Virasoro algebra by Proposition 3.2. This implies that U is isomorphic to $\bar{V}(c, 0)$ and

$$ch_q U = ch_q \bar{V}(c, 0) = \frac{q^{-c/24}}{\prod_{n>1}(1-q^n)}$$

which contradicts to Lemma 3.3.

So we can assume that $c = c_{p,q}$ for coprime p, q such that $1 < p < q$. In this case U is either isomorphic to $\bar{V}(c_{p,q}, 0)$ or $L(c_{p,q}, 0)$ as a module for the Virasoro algebra. Again by Lemma 3.3, U is not isomorphic to $\bar{V}(c_{p,q}, 0)$ by grading restriction. This forces U to be isomorphic to $L(c_{p,q}, 0)$, as desired. \square

A vertex operator algebra V^1 is called an *extension* of another vertex operator algebra V^2 if V^2 is isomorphic to a vertex operator subalgebra of V^1 with the same Virasoro

vector. In particular, V^1 is a V^2 -module. Note that if V^2 is rational then V^2 has only finitely many irreducible modules up to isomorphism and V^1 is necessarily a finite direct sum of irreducible V^2 -module due to finiteness of the homogeneous subspaces of vertex operator algebras.

We are now in a position to present the main result of this paper.

Theorem 3.5 *Let V be a simple, rational and C_2 -cofinite vertex operator algebra with central charge $c = \tilde{c} < 1$. Then V is an extension of $L(c_{p,q}, 0)$ by its simple modules for some coprime p, q such that $1 < p < q$.*

Proof: By Proposition 3.2, the vertex operator subalgebra U of V is rational. So as module for U V is a direct sum of finitely many irreducible U -modules. \square

So this theorem reduces the classification of simple, rational and C_2 -cofinite vertex operator algebras with central charge $c = \tilde{c} < 1$ to the problem of classification of extensions of vertex operator algebras $L(c_{p,q}, 0)$. It will certainly be a difficult and complicated task to construct the extensions of $L(c_{p,q}, 0)$ for all p, q . One definitely needs both coset construction and orbifold theory to achieve this. We will not go to this direction in this paper. See [LLY] for some extensions of $L(c_{p,p+1}, 0)$ by a simple module.

We remark that the condition that $c = \tilde{c}$ in Theorem 3.5 is necessary. Here is a counter example in which V is a simple, rational and C_2 -cofinite vertex operator algebra with $c < 1$, but V is not an extension of $L(c_{p,q}, 0)$ for any coprime integers $1 < p < q$. Recall that $L(c_{2,5}, 0)$ is a simple, rational and C_2 cofinite vertex operator algebra with central charge $-22/5$ and effective central charge $2/5$. Let W be any simple, rational and C_2 -cofinite vertex operator algebra whose central charge and effective central charge are 5. For example, one can take W to be a lattice vertex operator algebra V_L for a rank 5 positive definite even lattice L . Set $V = L(c_{2,5}, 0) \otimes W$. Then V is a simple, rational and C_2 -cofinite vertex operator algebra with central charge $c = 3/5$. Since any irreducible V -module is a tensor product of an irreducible $L(c_{2,5}, 0)$ -module with an irreducible W -module (cf. [FHL], [DMZ]), we see that the effective central charge of V is $\tilde{c} = 27/5$. Then $3/5$ is different from $c_{p,q}$ for all coprime integers p, q such that $1 < p < q$. In order to see this let $c_{p,q} = 3/5$. Then $15(p - q)^2 = pq$. Clearly, $q - p \neq 1$. Let d be a prime divisor of $q - p$. Then d is also a divisor of p or q . As a result d is a common divisor of p and q . This is a contradiction. This shows that V is not an extension of $L(c_{p,q}, 0)$ by finitely many irreducible modules for any such p, q .

Recall from [DM2] that for vertex operator algebra $L(c_{p,q}, 0)$, the effective central charge $\tilde{c} = 1 - \frac{6}{pq}$. So $c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq} = \tilde{c}$ if and only if $q - p = 1$, or if and only if $L(c_{p,q}, 0)$ is unitary. So our theorem eliminate the nonunitary rational Virasoro vertex operator algebras $L(c_{p,q}, 0)$ for $q \neq p + 1$ in our consideration. But one can still ask if any extension of $L(c_{p,q}, 0)$ for $q \neq p + 1$ satisfies the assumptions in Theorem 3.5. Note that that all the irreducible modules of $L(c_{p,q}, 0)$ are $L(c_{p,q}, h_{r,s})$ where

$$h_{r,s} = \frac{(sp - rq)^2 - (p - q)^2}{4pq}$$

for integers r, s in the ranges $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$. So each extension V of $L(c_{p,q}, 0)$ is a finite direct sum of irreducible $L(c_{p,q}, 0)$ -module and any irreducible V -module is also a finite direct sum of $L(c_{p,q}, h_{r,s})$'s. We suspect that the effective central charge $\tilde{c} = c$ for V if and only if $c = c_{p,p+1}$. But we do not have a proof for this.

We now turn our attention to an arbitrary simple vertex operator algebra V . We still denote by U the vertex operator subalgebra generated by the Virasoro vector ω . It has been a problem in the theory of vertex operator algebra whether U is a simple vertex operator algebra of U . Here is a solution.

Theorem 3.6 *If V is a simple, rational and C_2 -cofinite vertex operator algebra. Then the Virasoro vertex operator algebra U is simple if either $c \geq 1$ or $c = \tilde{c} < 1$.*

Proof: If $c \neq c_{p,q}$ for all p, q , $\bar{V}(c, 0)$ is simple (see Proposition 3.2 and must be isomorphic to U . If $c = c_{p,q}$, U is isomorphic to $L(c_{p,q}, 0)$ by Corollary 3.4. \square

We again remark that if $c \neq \tilde{c}$, Theorem 3.6 is not valid. In this case we consider $V = L(c_{2,5}, 0)^{\otimes 5} \otimes W$ where W is any simple, rational and C_2 -cofinite vertex operator algebra such that both the central charge and the effective central charge are 22. Then V is a simple, rational and C_2 -cofinite vertex operator algebra with central charge 0 and effective central charge 2. Let ω^1 and ω^2 be the Virasoro vectors of $L(c_{2,5}, 0)^{\otimes 5}$ and W , respectively. Then the Virasoro vector of V is $\omega = \omega^1 \otimes 1 + 1 \otimes \omega^2$. Clearly, $\omega \neq 0$. This implies that the vertex operator subalgebra U generated by ω is not equal to $\mathbb{C} = L(0, 0)$. So U is not simple.

References

- [AM] G. Anderson and G. Moore, Rationality in conformal field theory, *Comm. Math. Phys.* **117** (1988), 441-450.
- [A] Tom Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer-Verlag, New York, 1990.
- [DGL] C. Dong, R. Griess Jr. and C. Lam, Uniqueness results of the moonshine vertex operator algebra, *American Journal of Math.* **129** (2007), 583-609.
- [DLM1] C. Dong, H. Li and G. Mason, Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571-600.
- [DLM2] C. Dong, H. Li and G. Mason, Modular invariance of trace functions in orbifold theory and generalized moonshine, *Comm. Math. Phys.* **214** (2000), 1-56.
- [DLMM] C. Dong, H. Li, G. Mason and P. Montague, The radical of a vertex operator algebra, in: *Proc. of the Conference on the Monster and Lie algebras at The Ohio State University, May 1996*, ed. by J. Ferrar and K. Harada, Walter de Gruyter, Berlin- New York, 1998, 17-25. York, 1998.

- [DM1] C. Dong and G. Mason, Holomorphic vertex operator algebras of small central charges, *Pacific J. Math.* **213** (2004), 253-266.
- [DM2] C. Dong and G. Mason, Rational vertex operator algebras and the effective central charge, *International Math. Research Notices* **56** (2004), 2989-3008.
- [DMZ] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, *Proc. Symp. Pure. Math., American Math. Soc.* **56** II (1994), 295-316.
- [FF] B. Feigin and D. Fuchs, Verma Modules over the Virasoro Algebra, *Lect. Notes in Math.* **1060**, Springer, (1984), 230-245.
- [FHL] I. B. Frenkel, Y. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs American Math. Soc.* **104**, 1993.
- [FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, *Pure and Applied Math.*, Vol. **134**, Academic Press, 1988.
- [FZ] I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123-168.
- [FQS] D. Friedan, Z. Qiu and S. Shenker, Details of the non-unitarity proof for highest weight representations of Virasoro Algebra, *Comm. Math. Phys.* **107** (1986), 535-542.
- [GN] M. Gaberdiel and A. Neitzke, Rationality, quasirationality and finite W-algebras, *Comm. Math. Phys.* **238** (2003), 305-331.
- [GKO] P. Goddard, A. Kent and D. Olive, Unitary representations of the Virasoro Algebra and super-Virasoro algebras, *Comm. Math. Phys.* **103** (1986), 105-119.
- [KL] Y. Kawahigashi and R. Longo, Classification of local conformal nets. Case $c < 1$, *Ann. of Math.* **160** (2004), 493-522.
- [KM] M. Knopp and G. Mason, On vector-valued modular forms and their Fourier coefficients, *Acta Arith.* **110** (2003), 117-124.
- [LLY] C. Lam, N. Lam and H. Yamauchi, Extensions of unitary Virasoro vertex operator algebra by a simple module, *International Math. Research Notices* **2003** (2003), 577-611.
- [W] W. Wang, Rationality of Virasoro vertex operator algebras, *International Math. Research Notices*, **71** (1993), 197-211.
- [X] F. Xu, New braided endomorphisms from conformal inclusions, *Comm. Math. Phys.* **192** (1998), 347-403.

- [Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237-302.